THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 4

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1. State the definition of Riemann (Darboux) integrability using upper and lower integrals.

2. Show that a continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

3. Show that the set of all Riemann integrable functions $f : [a, b] \to \mathbb{R}$ form an infinite dimensional vector space over $\mathbb{R}$.

4. Show that if f is bounded on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is another function that equals to g except on a finite set, then g has the same upper and lower sums as f.

   In particular, if two bounded functions differ only at a finite set of points, then integrability of one function implies the other and they will have the same Riemann integrals if exist.

5. Find the upper and lower integrals for the following functions:

   (a) $f(x) := \chi_{\mathbb{Q}}$ on $[0, 1]$.

   (b) $f(x) := \begin{cases} 1, & \text{if} \quad x = \frac{1}{n} \text{ for some } n = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$

   These two functions both differ from the 0 function at countably many points, but they have different integrability properties.

   (Reference Only, for those taking MATH 4050) More generally, we have the following theorem:

   Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be bounded, and denote*

   $$D := \{x \in [a, b] : \ f \textit{ is discontinuous at } x\}.$$

   Then f is Riemann integrable on $[a, b]$ if and only if D has zero Lebesgue measure.

6. Show that a monotone function on $[a, b]$ is Riemann integrable, although it may have countably many discontinuities. (To sidetrack a little, it is a famous theorem that a monotone function can have at most countably many discontinuities.)